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Two-neutrino positron double-beta decay of ^{106}Cd for the $0^+ \longrightarrow 0^+$ transition

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Abstract. The two-neutrino positron double-beta decay of ^{106}Cd for the $0^+ \rightarrow 0^+$ transition has been studied in the Hartree-Fock-Bogoliubov model in conjunction with the summation method. In the first step, the reliability of the intrinsic wave functions of ^{106}Cd and ^{106}Pd nuclei has been tested by comparing the theoretically calculated results for yrast spectra, reduced $B(E2{:}0^+ \rightarrow 2^+)$ transition probabilities, quadrupole moments $Q(2^+)$ and gyromagnetic factors $g(2^+)$ with the available experimental data. In the second step, the nuclear transition matrix element $M_{2\nu}$ and the half-life $T_{1/2}^{2\nu}$ for the $0^+ \rightarrow 0^+$ transition have been calculated with these wave functions. Moreover, we have studied the effect of deformation on the nuclear transition matrix element $M_{2\nu}$.

PACS. 23.40.Hc Relation with nuclear matrix elements and nuclear structure – 21.60.Jz Hartree-Fock and random-phase approximations – 23.20.-g Electromagnetic transitions – 27.60.+j $90 \leq A \leq 149$

1 Introduction

The nuclear double-beta ($\beta\beta$) decay, one of the rarest processes of Nature, is characterized by two modes. They are the two-neutrino double-beta (2ν $\beta\beta$) decay and the neutrinoless double-beta (0ν $\beta\beta$) decay. These modes can be classified into double-electron ($\beta^-\beta^-$) emission, double-positron ($\beta^+\beta^+$) emission, electron-positron conversion (β^+EC) and double electron capture (ECEC). The latter three processes are energetically competing and we shall refer to them as positron double-beta decay (e^+DBD) modes. If the 0ν $\beta\beta$ decay were observed, the e^+DBD processes would play a crucial role in discriminating the finer issues like dominance of Majorana neutrino mass or the right-handed current. The theoretical implications and experimental aspects of e^+DBD modes have been widely reviewed over the past years [1–8].

The half-lives of many $\beta^-\beta^-$ emitters are shorter, compared with the other modes, due to a larger available phase space. For this reason they were the natural choice for the experimental observation to start with. However, the experimental sensitivity of the $\beta^-\beta^-$ decay mode gets limited because of the presence of electron background. On the other hand, from the experimental point of view, the e^+DBD modes are relatively easier to be separated from the background contaminations. Moreover, the e^+DBD modes are also attractive due to the possibility to detect the coincidence signals from four γ-rays, two γ-rays and one γ-ray for $\beta^+\beta^+$, β^+EC, ECEC modes, respectively. The Q value for the 2ν ECEC mode can be large enough (up to 2.8 MeV) but the detection of the $0^+ \rightarrow 0^+$ transition is difficult since only X-rays are emitted.

There have been very few experimental attempts for determining the half-lives of 2ν e^+DBD modes even for the best candidate ^{106}Cd [9–16] but one of the latest observations is very close to the predictions for the β^+EC mode [14]. With improved sensitivity in detection systems of the planned bigger Osaka-OTO experiment [17], it is expected that 2ν e^+DBD modes will be in an observable range in the near future [18]. Hence, a timely reliable prediction of the half-life of ^{106}Cd decay will be helpful in designing of an experimental set-up and analysis of data.

Rosen and Primakoff were the first to study the 2ν e^+DBD modes theoretically [2]. Later on, Kim and Kubodera estimated the half-lives of all the three modes with modified nuclear transition matrix elements (NTMEs) and non-relativistic phase space factors [19]. Abad *et al.* performed similar calculations using relativistic Coulomb wave functions [20]. In the meantime, the QRPA emerged as the most successful model in explaining the quenching of NTMEs by incorporating the particle-particle part of the effective nucleon-nucleon interaction in the

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proton-neutron channel and the observed $T_{1/2}^{2\nu}$ of several 2ν $\beta\beta$ decay emitters were reproduced successfully [6]. Staudt *et al.* used the QRPA model for evaluating 2ν $\beta^+\beta^+$ decay transition matrix elements [21]. Subsequently, the 2ν e^+DBD modes were studied in QRPA and its extensions [13,22–26], $SU(4)_{\sigma\tau}$ [27] and SSDH [28].

A vast amount of data concerning the level energies as well as electromagnetic properties have been compiled through experimental studies involving in-beam γ-ray spectroscopy over the past years. Hence, there is no need to study the $\beta\beta$ decay as an isolated nuclear process. The availability of data permits a rigorous and detailed critique of the ingredients of the microscopic framework that seeks to provide a description of nuclear $\beta\beta$ decay. However, most of the calculations of e^+DBD transition matrix elements performed so far but for the work of Barabash *et al.* [13] and Suhonen *et al.* [25] do not fully satisfy this criterion.

The nuclear structure in the mass region $A = 100$ offers a nice example of shape transition, *i.e.* a sudden onset of deformation at neutron number $N = 60$. The nuclei are soft vibrators for $N < 60$ and quasirotors for $N > 60$. The nuclei with neutron number $N = 60$ are transitional nuclei. Hence, it is expected that deformation degrees of freedom will play some crucial role in the structure of ^{106}Pd and ^{106}Cd nuclei. Further, the pairing of like nucleons plays an important role in all $\beta\beta$ decay emitters, which are even-Z and even-N nuclei. Hence, it is desirable to have a framework for the study of $\beta\beta$ decay in which the pairing and deformation degree of freedom are treated on equal footing in its formalism. The Projected Hartree-Fock-Bogoliubov (PHFB) model is a very reasonable choice which fulfills these requirements. The successful study of shape transition vis-à-vis electromagnetic properties of various nuclei in the PHFB model [29–32] using pairing–plus–quadrupole-quadrupole (PPQQ) [33] interaction motivates us to apply the HFB wave functions to study the 2ν e^+DBD modes of ^{106}Cd.

Further, it has been shown that there exists an inverse correlation between the Gamow-Teller strength and the quadrupole moment [34,35]. It is well known that the pairing degree of freedom accounts for the preference of nuclei to have a spherical form, whereas the quadrupole-quadrupole (QQ) interaction increases the collectivity in the nuclear intrinsic wave functions and makes the nucleus deformed. Hence, the PHFB model using the PPQQ interaction is a convenient choice to examine the explicit role of deformation on NTME $M_{2\nu}$.

Our aim is to study the 2ν e^+DBD transition of $^{106}\text{Cd} \to {}^{106}\text{Pd}$ for the $0^+ \to 0^+$ transition together with other observed nuclear properties using the PHFB model. In the PHFB model, the NTME $M_{2\nu}$ is usually calculated using the closure approximation. In the present calculation, we have avoided the closure approximation by making use of the summation method [36]. In sect. 2, we briefly outline the theoretical framework. In sect. 3, the reliability of the wave functions is first established by calculating the yrast spectra, reduced $B(E2{:}0^+ \to 2^+)$ transition probabilities, static quadrupole moments $Q(2^+)$ and g-factors $g(2^+)$ of both parent ^{106}Cd and daughter ^{106}Pd nuclei and by comparing them with the available experimental data. The half-lives of 2ν e^+DBD modes for the $0^+ \to 0^+$ transition have been given as prediction. The role of deformation on NTME $M_{2\nu}$ has also been studied. We present the conclusions in sect. 4.

2 Theoretical framework

The theoretical formalism to calculate the half-lives of 2ν e^+DBD modes has been given by Doi *et al.* [5] and Suhonen *et al* [6]. Hence, we briefly outline the steps of the above derivations for clarity in notation following Doi *et al.* [5]. Details of the mathematical expressions used to calculate electromagnetic properties are given by Dixit *et al.* [37].

The half-life of the 2ν e^+DBD mode for the $0^+ \to 0^+$ transition is given by

$$\left[T_{1/2}^{2\nu}(0^+ \to 0^+)\right]^{-1} = G_{2\nu}\left|M_{2\nu}\right|^2 , \tag{1}$$

where the integrated kinematical factor $G_{2\nu}$ can be calculated with good accuracy [5] and the NTME $M_{2\nu}$ is given by

$$M_{2\nu} = \sum_N \frac{\langle 0_F^+||\boldsymbol{\sigma}\tau^-||1_N^+\rangle\langle 1_N^+||\boldsymbol{\sigma}\tau^-||0_I^+\rangle}{E_N - (E_I + E_F)/2} \tag{2}$$

$$= \sum_N \frac{\langle 0_F^+||\boldsymbol{\sigma}\tau^-||1_N^+\rangle\langle 1_N^+||\boldsymbol{\sigma}\tau^-||0_I^+\rangle}{E_0 + E_N - E_I} , \tag{3}$$

where

$$E_0 = \frac{1}{2}\left(E_I - E_F\right) = \frac{1}{2}Q_{\beta\beta} + m_e = \frac{1}{2}W_0 . \tag{4}$$

Here, W_0 is the total energy released and is given by

$$W_0 = E_I - E_F , \tag{5}$$

$$W_0(\beta^+\beta^+) = Q_{\beta^+\beta^+} + 2m_e , \tag{6}$$

$$W_0(\beta^+\text{EC}) = Q_{\beta^+\text{EC}} + e_b , \tag{7}$$

$$W_0(\text{ECEC}) = Q_{\text{ECEC}} - 2m_e + e_{b1} + e_{b2} . \tag{8}$$

The summation over intermediate states can be completed using the summation method [36] and the $M_{2\nu}$ can be written as

$$M_{2\nu} = \frac{1}{E_0}\left\langle 0_F^+ \left| \sum_m (-1)^m \Gamma_{-m} F_m \right| 0_I^+ \right\rangle , \tag{9}$$

where the Gamow-Teller (GT) operator Γ_m is given by

$$\Gamma_m = \sum_s \sigma_{ms}\tau_s^- \tag{10}$$

and

$$F_m = \sum_{\lambda=0}^{\infty} \frac{(-1)^\lambda}{E_0^\lambda} D_\lambda \Gamma_m \tag{11}$$

with

$$D_\lambda \Gamma_m = [H, [H, \ldots, [H, \Gamma_m] \ldots]]^{(\lambda \text{ times})} . \quad (12)$$

In the present work, we use a Hamiltonian with PPQQ interaction type [33] of the effective two-body interaction. Explicitly, the Hamiltonian is written as

$$H = H_{\rm sp} + V(P) + \chi_{qq} V(QQ) , \quad (13)$$

where $H_{\rm sp}$ denotes the single-particle Hamiltonian. The pairing part of the effective two-body interaction $V(P)$ is written as

$$V(P) = -\left(\frac{G}{4}\right) \sum_{\alpha\beta} (-1)^{j_\alpha + j_\beta - m_\alpha - m_\beta} a^\dagger_\alpha a^\dagger_{\bar\alpha} a_{\bar\beta} a_\beta , \quad (14)$$

where α denotes the quantum numbers $(nljm)$. The state $\bar\alpha$ is the same as α but with the sign of m reversed. The QQ part of the effective interaction $V(QQ)$ is given by

$$V(QQ) = -\left(\frac{\chi}{2}\right) \sum_{\alpha\beta\gamma\delta} \sum_\mu (-1)^\mu \langle\alpha|q^2_\mu|\gamma\rangle \langle\beta|q^2_{-\mu}|\delta\rangle a^\dagger_\alpha a^\dagger_\beta a_\delta a_\gamma, \quad (15)$$

where

$$q^2_\mu = \left(\frac{16\pi}{5}\right)^{1/2} r^2 Y^2_\mu(\theta, \phi) . \quad (16)$$

The χ_{qq} is an arbitrary parameter and the final results are obtained by setting the $\chi_{qq} = 1$. The purpose of introducing χ_{qq} is to study the role of deformation by varying the strength of the QQ interaction.

When the GT operator commutes with the effective two-body interaction, eq. (12) can be further simplified to

$$M_{2\nu} = \sum_{\pi,\nu} \frac{\langle 0^+_F || \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} \tau^- \tau^- || 0^+_I \rangle}{E_0 + \varepsilon(n_\nu, l_\nu, j_\nu) - \varepsilon(n_\pi, l_\pi, j_\pi)} . \quad (17)$$

In the case of the pseudo-$SU(3)$ model [38–40], the GT operator commutes with the two-body interaction and the energy denominator is a well-defined quantity without any free parameter. It has been evaluated exactly for the 2ν $\beta^-\beta^-$ [38,39] and 2ν ECEC modes [40] in the context of the pseudo-$SU(3)$ scheme. However, in the present case, the model Hamiltonian is not isospin symmetric. Hence, the energy denominator is not as simple as in eq. (17). But the violation of isospin symmetry for the QQ part of our model Hamiltonian is negligible, as will be evident from the parameters of the two-body interaction given later. And the violation of isospin symmetry for the pairing part of the two-body interaction is presumably small. Under these assumptions, the expression to calculate the NTME $M_{2\nu}$ of the e^+DBD modes for the $0^+ \to 0^+$ transition in the PHFB model is obtained as follows.

The essential idea behind the HFB theory is to transform particle coordinates to quasiparticle coordinates through a general Bogoliubov transformation such that the quasiparticles are relatively weakly interacting. Essentially, the Hamiltonian H is expressed as

$$H = E_0 + H_{\rm qp} + H_{\text{qp-int}} \quad (18)$$

where E_0 is the energy of the quasiparticle vacuum, $H_{\rm qp}$ is the elementary quasiparticle excitations and $H_{\text{qp-int}}$ is a weak interaction between the quasiparticles. In the HFB theory, the interaction between the quasiparticles is usually neglected and the Hamiltonian H is approximated by an independent quasiparticle Hamiltonian. In the time-dependent HFB (TDHFB) model or the quasiparticle random phase approximation (QRPA), some effects of quasiparticle interaction can be included. The axially symmetric intrinsic HFB state with $K = 0$ can be written as

$$|\Phi_0\rangle = \prod_{im} (u_{im} + v_{im} b^\dagger_{im} b^\dagger_{i\bar m}) |0\rangle , \quad (19)$$

where the creation operators $b^\dagger_{im}$ and $b^\dagger_{i\bar m}$ are given by

$$b^\dagger_{im} = \sum_\alpha C_{i\alpha,m} a^\dagger_{\alpha m} \quad \text{and}$$
$$b^\dagger_{i\bar m} = \sum_\alpha (-1)^{l+j-m} C_{i\alpha,m} a^\dagger_{\alpha,-m} . \quad (20)$$

Using the standard projection technique, a state with good angular momentum $\mathbf{J}$ is obtained from the HFB intrinsic state through the following relation:

$$|\Psi^J_{MK}\rangle = P^J_{MK} |\Phi_K\rangle = \left[\frac{(2J+1)}{8\pi^2}\right] \int D^J_{MK}(\Omega) R(\Omega) |\Phi_K\rangle {\rm d}\Omega , \quad (21)$$

where $R(\Omega)$ and $D^J_{MK}(\Omega)$ are the rotation operator and the rotation matrix, respectively.

Finally, one obtains the following expression for the NTMEs of the e^+DBD modes:

$$\begin{aligned} M_{2\nu} &= \sum_{\pi,\nu} \frac{\langle \Psi^{J_f=0}_{00} || \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} \tau^- \tau^- || \Psi^{J_i=0}_{00} \rangle}{E_0 + \varepsilon(n_\nu, l_\nu, j_\nu) - \varepsilon(n_\pi, l_\pi, j_\pi)} \\ &= [n^{J_f=0}_{Z-2,N+2} n^{J_i=0}_{Z,N}]^{-1/2} \int_0^\pi n_{(Z,N),(Z-2,N+2)}(\theta) \\ &\times \sum_{\alpha\beta\gamma\delta} \frac{\langle \alpha\beta | \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \tau^- \tau^- | \gamma\delta \rangle}{E_0 + \varepsilon_\alpha(n_\nu, l_\nu, j_\nu) - \varepsilon_\gamma(n_\pi, l_\pi, j_\pi)} \\ &\times \sum_{\varepsilon\eta} \left[\left(1 + F^{(\nu)}_{Z,N}(\theta) f^{(\nu)*}_{Z-2,N+2}\right)\right]^{-1}_{\varepsilon\alpha} (f^{(\nu)*}_{Z-2,N+2})_{\varepsilon\beta} \\ &\times \left[\left(1 + F^{(\pi)}_{Z,N}(\theta) f^{(\pi)*}_{Z-2,N+2}\right)\right]^{-1}_{\gamma\eta} (F^{(\pi)*}_{Z,N})_{\eta\delta} \sin\theta {\rm d}\theta , \end{aligned} \quad (22)$$

where

$$n^J = \int_0^\pi \{\det[1 + F^{(\pi)}(\theta) f^{(\pi)\dagger}]\}^{1/2} \times \{\det[1 + F^{(\nu)}(\theta) f^{(\nu)\dagger}]\}^{1/2} d^J_{00}(\theta) \sin(\theta) {\rm d}\theta \quad (23)$$

and

$$n_{(Z,N),(Z-2,N+2)}(\theta) = \{\det[1 + F^{(\pi)}_{Z,N}(\theta) f^{(\pi)\dagger}_{Z-2,N+2}]\}^{1/2} \times \{\det[1 + F^{(\nu)}_{Z,N}(\theta) f^{(\nu)\dagger}_{Z-2,N+2}]\}^{1/2} \quad (24)$$

Table 1. Variation in intrinsic quadrupole moment $\langle Q_0^2 \rangle$ and excitation energies (in MeV) of $J^\pi = 2^+$, 4^+, and 6^+ yrast states of ^{106}Cd and ^{106}Pd nuclei with change in χ_{pn}, keeping fixed $G_p = 30/A$ MeV, $G_n = 20/A$ MeV, $\chi_{pp} = \chi_{nn} = 0.0105$ MeV b^{-4} and $\varepsilon(0h_{11/2}) = 8.6$ MeV.

Nucleus	Theo.						Expt. [46]
	$\chi_{pn} =$	0.0142	0.0145	0.0148	0.0151	0.0154	
^{106}Cd	$\langle Q_0^2 \rangle$	43.3772	44.7355	46.0289	47.3807	49.0039	
	E_{2^+}	0.7749	0.7339	0.6797	**0.6220**	0.5869	**0.6327**
	E_{4^+}	1.9024	1.8728	1.8022	1.7129	1.6690	1.4939
	E_{6^+}	3.2993	3.3029	3.2389	3.1411	3.1089	2.4918
^{106}Pd	$\langle Q_0^2 \rangle$	51.4360	52.4295	53.4325	54.2709	55.1674	
	E_{2^+}	0.5524	**0.5036**	0.4819	0.4500	0.4415	**0.5119**
	E_{4^+}	1.5706	1.4668	1.4269	1.3554	1.3435	1.2292
	E_{6^+}	2.8526	2.7089	2.6655	2.5652	2.5620	2.0766

with

$$[F_{Z,N}(\theta)]_{\alpha\beta} = \sum_{m'_\alpha m'_\beta} d^{j_\alpha}_{m_\alpha,m'_\alpha}(\theta) d^{j_\beta}_{m_\beta,m'_\beta}(\theta) f_{j_\alpha m'_\alpha, j_\beta m'_\beta} \tag{25}$$

and

$$[f_{Z,N}]_{\alpha\beta} = \sum_i C_{ij_\alpha,m_\alpha} C_{ij_\beta,m_\beta} (v_{im_\alpha}/u_{im_\alpha}) \delta_{m_\alpha,-m_\beta} . \tag{26}$$

Here π (ν) stands for the proton (neutron) of nuclei involved in 2ν e^+DBD. The results of PHFB calculations which are summarized by the amplitudes (u_{im}, v_{im}) and the expansion coefficients $C_{ij,m}$ are used to set up the matrices for $[F_{Z,N}(\theta)]_{\alpha\beta}$ and $[f_{Z,N}]_{\alpha\beta}$ given by eqs. (25) and (26), respectively. Finally, the required NTME $M_{2\nu}$ is calculated in a straightforward manner using eq. (22) with 20-point Gaussian quadrature in the range (0, π).

3 Results and discussions

The model space, single-particle energies (SPEs) and two-body interactions are the same as our earlier calculation on the 2ν $\beta\beta$ decay of ^{100}Mo for the $0^+ \to 0^+$ transition [37]. We include a brief discussion of them in the following for convenience. We have treated the doubly even nucleus ^{76}Sr ($N = Z = 38$) as an inert core with the valence space spanned by the orbits $1p_{1/2}$, $2s_{1/2}$, $1d_{3/2}$, $1d_{5/2}$, $0g_{7/2}$, $0g_{9/2}$ and $0h_{11/2}$ for protons and neutrons. The orbit $1p_{1/2}$ has been included in the valence space to examine the role of the $Z = 40$ proton core vis-à-vis the onset of deformation in the highly neutron-rich isotopes.

The set of single-particle energies (SPEs) used here is (in MeV) $\varepsilon(1p_{1/2}) = -0.8$, $\varepsilon(0g_{9/2}) = 0.0$, $\varepsilon(1d_{5/2}) = 5.4$, $\varepsilon(2s_{1/2}) = 6.4$, $\varepsilon(1d_{3/2}) = 7.9$, $\varepsilon(0g_{7/2}) = 8.4$ and $\varepsilon(0h_{11/2}) = 8.6$ for proton and neutrons. This set of SPEs but for the $\varepsilon(0h_{11/2})$, which is slightly lowered, has been employed in a number of successful shell model [41,42] as well as variational-model [29–32] calculations for nuclear properties in the mass region $A = 100$. The strengths of the pairing interaction are fixed through the relations $G_p = 30/A$ MeV and $G_n = 20/A$ MeV, which are the same used by Heestand *et al.* [43] to explain the experimental $g(2^+)$ data of some even-even Ge, Se, Mo, Ru, Pd, Cd and Te isotopes in Greiner's collective model [44]. The strengths of the like-particle components of the QQ interaction are taken as: $\chi_{pp} = \chi_{nn} = 0.0105$ MeV b^{-4}, where b is the oscillator parameter. The strength of the proton-neutron (pn) component of the QQ interaction χ_{pn} is varied so as to reproduce the experimentally observed excitation energy of the 2^+ state E_{2^+} of ^{106}Cd and ^{106}Pd as closely as possible. The χ_{pn} has been fixed to be 0.0151 and 0.0145 MeV b^{-4} for ^{106}Cd and ^{106}Pd, respectively. Thus, for a given model space, SPEs, G_p, G_n and χ_{pp}, we have fixed χ_{pn} through the experimentally available energy spectra. These values for the strength of the QQ interaction are comparable to those suggested by Arima on the basis of an empirical analysis of the effective two-body interactions [45].

We have varied the χ_{pn} to obtain the yrast spectra of ^{106}Cd and ^{106}Pd in optimum agreement with experimental results [46]. We have taken the theoretical spectra to be the optimum if the excitation energy of the 2^+ state E_{2^+} is reproduced as closely as possible in comparison to the experimental results. Theoretically calculated intrinsic quadrupole moments $\langle Q_0^2 \rangle$ and yrast energies for the E_{2^+}-to-E_{6^+} levels of ^{106}Cd and ^{106}Pd for $\chi_{pn} = 0.0142$ to 0.0154 are presented in table 1. In the case of ^{106}Cd, $\langle Q_0^2 \rangle$ increases by 5.6267 units and E_{2^+} decreases by 0.1880 MeV as χ_{pn} is varied from 0.0142 to 0.0154 MeV b^{-4}. For the same variation in χ_{pn}, $\langle Q_0^2 \rangle$ increases by 3.7314 units and E_{2^+} decreases by 0.1109 MeV in the case of ^{106}Pd. This observed inverse correlation between $\langle Q_0^2 \rangle$ and E_{2^+} is understandable as there is an enhancement in the collectivity of the intrinsic state with the increase of $|\chi_{pn}|$, E_{2^+} decreases. This is known as Grodzins's rule [47]. The theoretically calculated E_{2^+} for ^{106}Cd is 0.6220 MeV corresponding to $\chi_{pn} = 0.0151$ MeV b^{-4} in comparison to the experimentally observed value 0.6327 MeV. In case of ^{106}Pd, the theoretically calculated E_{2^+} for $\chi_{pn} = 0.0145$ MeV b^{-4} is 0.5036 MeV in comparison to the observed value of 0.5119 MeV. All these input parameters are kept fixed for the calculation of spectroscopic properties as well as the NTMEs discussed below.

Table 2. Comparison of the calculated and experimentally observed reduced transition probabilities $B(E2: 0^+ \to 2^+)$, static quadrupole moments $Q(2^+)$ and g-factors $g(2^+)$. Here $B(E2)$ and $Q(2^+)$ are calculated in units of e^2 b^2 and e b, respectively, for effective charge $e_p = 1 + e_{\rm eff}$ and $e_n = e_{\rm eff}$. $g(2^+)$ has been calculated in units of nuclear magneton for $g_l^\pi = 1.0$, $g_l^\nu = 0.0$ and $g_s^\pi = g_s^\nu = 0.60$. Corresponding references for experimentally observed values are given in parentheses.

Nucleus	$B(E2{:}0^+ \to 2^+)$				$Q(2^+)$				$g(2^+)$	
	Theo. $e_{\rm eff}$			Expt. [48]	Theo. $e_{\rm eff}$			Expt. [49]	Theo.	Expt. [49]
	0.40	0.50	0.60		0.40	0.50	0.60			
^{106}Cd	0.334	0.426	0.531	0.410±0.020 0.386±0.05	−0.52	−0.59	−0.66	−0.28±0.08	0.370	0.40±0.10
^{106}Pd	0.407	0.520	0.657	0.610±0.090 0.656±0.035	−0.58	−0.65	−0.73	−0.56±0.08 −0.51±0.08	0.466	0.398±0.021 0.30±0.06

Table 3. Experimental limits on half-lives $T_{1/2}^{2\nu}(0^+ \to 0^+)$, theoretically calculated $M_{2\nu}$ and corresponding $T_{1/2}^{2\nu}(0^+ \to 0^+)$ for the 2ν $\beta^+\beta^+$, $2\nu\beta^+$EC and 2ν ECEC modes of ^{106}Cd. The numbers corresponding to a) and b) are calculated for $g_A = 1.261$ and 1.0, respectively.

Decay mode	Experiment		Theory				
	Ref.	$T_{1/2}^{2\nu}$ (y)	Ref.	Models	$\lvert M_{2\nu}\rvert$		$T_{1/2}^{2\nu}$ (y)
$\beta^+\beta^+$	[15]	$> 5.0 \times 10^{18}$	Present	PHFB	0.081	a)	**307.58×10^{25}**
	[14]	$> 2.4 \times 10^{20**}$				b)	**777.71×10^{25}**
	[13]	$> 1.0 \times 10^{19*}$	[26]	SQRPA(l.b.)	0.61	a)	5.38×10^{25}
	[12]	$> 9.2 \times 10^{17}$				b)	13.60×10^{25}
	[10]	$> 5.0 \times 10^{17}$		SQRPA(s.b.)	0.57	a)	6.16×10^{25}
	[9]	$> 2.6 \times 10^{17*}$				b)	15.58×10^{25}
			[25]	QRPA(WS)	0.166	a)	72.71×10^{25}
						b)	183.84×10^{25}
				QRPA(AWS)	0.722	a)	3.84×10^{25}
						b)	9.72×10^{25}
			[13]	QRPA(WS)	0.840	a)	2.84×10^{25}
						b)	7.18×10^{25}
				QRPA(AWS)	0.780	a)	3.29×10^{25}
						b)	8.33×10^{25}
			[23]	QRPA	0.218	a)	42.2×10^{25}
						b)	106.6×10^{25}
			[21]	QRPA			4.94×10^{25}

The calculated as well as the experimentally observed values of the reduced $B(E2{:}0^+ \to 2^+)$ transition probabilities, static quadrupole moments $Q(2^+)$, and the gyromagnetic factors $g(2^+)$ have been presented in table 2. We have calculated $B(E2)$ values for effective charges $e_{\rm eff} = 0.40$, 0.50, and 0.60, which are displayed in columns 2 to 4, respectively. The experimentally observed values are displayed in column 5. It is noticed that the calculated and the observed $B(E2)$ values [48] are in excellent agreement for $e_{\rm eff} = 0.5$. The theoretically calculated $Q(2^+)$ are tabulated in columns 6 to 8 for the same effective charges as given above. The experimental $Q(2^+)$ results [49] are given in column 9. It can be seen that for the same effective charge 0.5, the calculated values are close to the experimental limit in case of ^{106}Pd, while the agreement between the calculated and experimental values is off for ^{106}Cd. The $g(2^+)$ values are calculated with $g_l^\pi = 1.0$, $g_l^\nu = 0.0$, and $g_s^\pi = g_s^\nu = 0.60$. The calculated $g(2^+)$ is 0.370 nm and 0.466 nm for ^{106}Cd and ^{106}Pd, respectively. The theoretically calculated and experimentally observed $g(2^+)$ values are in good agreement for ^{106}Cd and slightly off by 0.047 nm for ^{106}Pd from the upper limit given by Raghavan [49]. The overall agreement between the calculated and observed electromagnetic properties of ^{106}Cd and ^{106}Pd suggests that the PHFB wave functions generated by fixing χ_{pn} to reproduce the yrast spectra are quite reliable.

The 2ν e^+DBD modes of ^{106}Cd for the $0^+ \to 0^+$ transition has been investigated by very few experimental groups, whereas some theoretical investigations have been made using the QRPA and its extensions [13,21–26], $SU(4)_{\sigma\tau}$ [27] and SSDH [28]. In table 3, we have compiled all the available experimental [9–16] and theoretical results [13,21–28] along with our calculated $M_{2\nu}$ and corresponding half-life $T_{1/2}^{2\nu}$. We have used phase space factors $G_{2\nu} = 4.991 \times 10^{-26}$ y^{-1}, 1.970×10^{-21} y^{-1} and 1.573×10^{-20} y^{-1} for the 2ν $\beta^+\beta^+$, 2ν β^+EC and 2ν ECEC modes, respectively, as given by Doi *et al.* [5]. The phase space integral has been evaluated for $g_A = 1.261$ by Doi *et al.* [5]. However, in heavy nuclei it is more justified to

Table 3. Continued.

Decay mode	Experiment		Theory				
	Ref.	$T_{1/2}^{2\nu}$ (y)	Ref.	Models	$\|M_{2\nu}\|$		$T_{1/2}^{2\nu}$ (y)
β^+EC	[15]	$> 1.2 \times 10^{18}$	Present	PHFB	0.081	a)	$\mathbf{77.925 \times 10^{21}}$
	[14]	$> 4.1\times10^{20}$				b)	$\mathbf{197.03 \times 10^{21}}$
	[13]	$> 0.66\times10^{19*}$	[26]	SQRPA(l.b.)	0.61	a)	1.36×10^{21}
	[12]	$> 2.6 \times 10^{17}$				b)	3.44×10^{21}
	[9]	$> 5.7 \times 10^{17*}$		SQRPA(s.b.)	0.57	a)	1.56×10^{21}
						b)	3.94×10^{21}
			[25]	QRPA(WS)	0.168	a)	17.99×10^{21}
						b)	45.48×10^{21}
				QRPA(AWS)	0.718	a)	0.98×10^{21}
						b)	2.49×10^{21}
			[27]	$SU(4)_{\sigma\tau}$	0.1947	a)	13.39×10^{21}
						b)	33.86×10^{21}
			[24]	RQRPA(WS)	0.550	a)	1.68×10^{21}
						b)	4.24×10^{21}
			[24]	RQRPA(AWS)	0.560	a)	1.62×10^{21}
						b)	4.09×10^{21}
			[13]	QRPA(WS)	0.840	a)	0.72×10^{21}
						b)	1.82×10^{21}
				QRPA(AWS)	0.780	a)	0.83×10^{21}
						b)	2.11×10^{21}
			[23]	QRPA	0.352	a)	4.1×10^{21}
						b)	10.4×10^{21}
			[22]	QRPA(WS)	0.493–0.660	a)	$(2.09 - 1.16) \times 10^{21}$
						b)	$(5.28 - 2.95) \times 10^{21}$
ECEC	[15]	$> 5.8 \times 10^{17}$	Present	PHFB	0.081	a)	$\mathbf{97.593 \times 10^{20}}$
	[16]	$> 1.0\times10^{18}$				b)	$\mathbf{246.76 \times 10^{20}}$
	[11]	$> 5.8\times10^{17}$	[26]	SQRPA(l.b.)	0.61	a)	2.6×10^{20}
						b)	6.57×10^{20}
				SQRPA(s.b.)	0.57	a)	1.96×10^{20}
						b)	4.96×10^{20}
			[25]	QRPA(WS)	0.168	a)	22.52×10^{20}
						b)	56.95×10^{20}
				QRPA(AWS)	0.718	a)	1.23×10^{20}
						b)	3.12×10^{20}
			[28]	SSDH(Theo)	0.280	a)	8.11×10^{20}
						b)	20.50×10^{20}
				SSDH(Exp)	0.170	a)	22.00×10^{20}
						b)	55.62×10^{20}
			[27]	$SU(4)_{\sigma\tau}$	0.1947	a)	16.77×10^{20}
						b)	42.40×10^{20}
			[24]	RQRPA(WS)	0.550	a)	2.10×10^{20}
						b)	5.31×10^{20}
			[24]	RQRPA(AWS)	0.560	a)	2.03×10^{20}
						b)	5.13×10^{20}
			[13]	QRPA(WS)	0.840	a)	0.90×10^{20}
						b)	2.28×10^{20}
				QRPA(AWS)	0.780	a)	1.05×10^{20}
						b)	2.64×10^{20}
			[23]	QRPA	0.270	a)	8.7×10^{20}
						b)	22.1×10^{20}
			[22]	QRPA(WS)	0.493–0.660	a)	$(2.62 - 1.46) \times 10^{20}$
						b)	$(6.61 - 3.69) \times 10^{20}$

* and ** denote the half-life limit for the $0\nu + 2\nu$ and $0\nu + 2\nu + 0\nu M$ modes, respectively.

Table 4. Effect of the variation in χ_{qq} on $\langle Q_0^2 \rangle$ and $M_{2\nu}$.

χ_{qq}	^{106}Cd			^{106}Pd			$\|M_{2\nu}\|$
	$\langle Q_0^2 \rangle_\pi$	$\langle Q_0^2 \rangle_\nu$	$\langle Q_0^2 \rangle$	$\langle Q_0^2 \rangle_\pi$	$\langle Q_0^2 \rangle_\nu$	$\langle Q_0^2 \rangle$	
0.00	0.0	0.0	0.0	0.0	0.0	0.0	0.1689
0.05	−0.0025	0.0039	0.0015	−0.0008	0.0057	0.0048	0.1709
0.20	−0.0087	0.0169	0.0082	0.1067	0.2100	0.3168	0.1636
0.40	−0.0100	0.0442	0.0342	0.0099	0.0701	0.0800	0.1624
0.60	0.0218	0.1261	0.1479	0.0483	0.1617	0.2100	0.1655
0.70	0.0683	0.2193	0.2876	0.0892	0.2455	0.3347	0.1682
0.80	0.1416	0.3594	0.5010	0.4521	0.8958	1.3479	0.1713
0.85	0.2227	0.5053	0.7280	11.2116	18.4734	29.6850	0.1432
0.90	11.63	20.0514	31.6814	15.534	25.0116	40.0650	0.1218
0.95	14.9910	26.1956	41.1866	17.5444	29.7372	47.2816	0.0935
1.00	17.4655	29.9152	47.3807	19.2454	33.1840	52.4295	0.0807
1.05	22.3626	34.0684	56.4230	20.4735	35.9085	56.3820	0.0831
1.15	31.6509	38.4407	70.0915	22.9707	39.9774	62.9481	0.0638
1.20	33.9053	39.6519	73.5572	24.8922	41.7666	66.6589	0.0417

use the nuclear-matter value of g_A around 1.0. Hence, the theoretical $T_{1/2}^{2\nu}$ are calculated for $g_A = 1.0$ and 1.261. We have presented only the theoretical $T_{1/2}^{2\nu}$ for those models for which no direct or indirect information about $M_{2\nu}$ or $G_{2\nu}$ is available to us.

We have evaluated the NTMEs $M_{2\nu}$ as well as half-lives $T_{1/2}^{2\nu}$ using both the summation method [36] and closure approximation [3] and compared them. In the summation method, the calculated NTME $M_{2\nu}$ is 0.081. The corresponding half-lives of the 2ν $\beta^+\beta^+$, 2ν β^+EC and 2ν ECEC modes are 7.78×10^{27} (3.08×10^{27}) y, 1.97×10^{23} (7.79×10^{22}) y and 2.47×10^{22} (9.76×10^{21}) y for $g_A = 1.0$ (1.261), respectively. We use the following prescription to evaluate the energy denominator E_d for 2ν e^+DBD modes in the closure approximation.

$$E_\mathrm{d} = E_\mathrm{d}(\beta^-\beta^-) - W_0 \,, \tag{27}$$

where $E_\mathrm{d}(\beta^-\beta^-)$ is the energy denominator for the 2ν $\beta^-\beta^-$ mode. The $E_\mathrm{d}(\beta^-\beta^-)$ is given by [3]

$$E_\mathrm{d}(\beta^-\beta^-) = 1.12A^{1/2} \,. \tag{28}$$

The value of the closure energy E_d is equal to 9.7733 MeV. The calculated NTME $M_{2\nu}$ in closure approximation is found to be 0.119. The half-lives of the 2ν $\beta^+\beta^+$, 2ν β^+EC and 2ν ECEC modes are 3.58×10^{27} (1.42×10^{27}) y, 9.08×10^{22} (3.59×10^{22}) y and 1.14×10^{22} (4.50×10^{21}) y for $g_A = 1.0$ (1.261), respectively. The ratio of NTMEs $M_{2\nu}$ in the closure approximation and summation method is 2.16. In the calculation of half-lives $T_{1/2}^{2\nu}$, there is an additional contribution due to g_A^4, which varies from unity to 2.53. Hence, the difference in $M_{2\nu}$ in the closure approximation and summation method can be compensated by the renormalization of g_A. It is clear from the above discussions that the difference in the results of the summation method and closure approximation does not manifest so far, as the calculation of half-lives $T_{1/2}^{2\nu}$ is concerned, due to uncertainty in g_A as well as unavailability of experimentally observed half-lives.

In column 3 of table 3, we have presented the experimentally observed limits on half-lives $T_{1/2}^{2\nu}$. In comparison to the theoretically predicted $T_{1/2}^{2\nu}$, the present experimental limits for the $0^+ \to 0^+$ transition of ^{106}Cd are smaller by a factor of 10^{5-7} in the case of the 2ν $\beta^+\beta^+$ mode but are quite close for the 2ν β^+EC and 2ν ECEC modes. The half-life $T_{1/2}^{2\nu}$ calculated in the PHFB model using the summation method differs from all the existing calculations. The presently calculated NTME $M_{2\nu}$ is smaller than the recently given results in the QRPA(WS) model of Suhonen and Civitarese [25] by a factor of 2 approximately for all the three modes. The theoretical $M_{2\nu}$ values of the PHFB model and $SU(4)_{\sigma\tau}$ [27] again differ by a factor of 2 approximately for the 2ν β^+EC and 2ν ECEC modes. On the other hand, the $M_{2\nu}$ calculated in our PHFB model is smaller than the values of Hirsch *et al.* [23] by a factor of 3 approximately in the case of the 2ν $\beta^+\beta^+$ and 2ν ECEC modes, while for the 2ν β^+EC mode the results differ by a factor of 4 approximately. All the rest of the calculations predict NTMEs which are larger than our predicted $M_{2\nu}$ approximately by a factor of 7 [22,24] to 10 [13].

We have studied the role of deformation on $\langle Q_0^2 \rangle$ and $M_{2\nu}$ vis-à-vis the variation of the strength of the *pn* part of the *QQ* interaction χ_{qq}. The results are tabulated in table 4. $\langle Q_0^2 \rangle$ of ^{106}Cd and ^{106}Pd remain almost constant as χ_{qq} is varied from 0.0 to 0.80. $M_{2\nu}$ also remains almost constant as χ_{qq} is changed from 0.0 to 0.80. As χ_{qq} is further changed from 0.80 to 1.20, $\langle Q_0^2 \rangle$ increases while $M_{2\nu}$ decreases to 0.0417 having a fluctuation at 1.05. To quantify the effect of deformation on $M_{2\nu}$, we define a quantity $D_{2\nu}$ as the ratio of $M_{2\nu}$ at zero deformation ($\chi_{qq} = 0$) and full deformation ($\chi_{qq} = 1$). $D_{2\nu}$ is given by

$$D_{2\nu} = \frac{M_{2\nu}(\chi_{qq} = 0)}{M_{2\nu}(\chi_{qq} = 1)} \,. \tag{29}$$

The value of $D_{2\nu}$ is 2.09, which suggests that $M_{2\nu}$ is quenched by a factor of approximately 2 due to deformation effects.

It is evident from the above discussions that it is difficult to establish the validity of different nuclear models presently employed to study 2ν e^+DBD due to limiting values in experimental results as well as uncertainty in g_A. Further work is necessary both in the experimental as well as the theoretical front to judge the relative applicability, success and failure of various models used so far for the study of 2ν e^+DBD processes before they can have better predictive power for the 0ν e^+DBD modes.

4 Conclusions

We have tested the quality of HFB wave functions by comparing the theoretically calculated results for a number of spectroscopic properties namely yrast spectra, reduced $B(E2{:}0^+ \to 2^+)$ transition probabilities, quadrupole moments $Q(2^+)$ and g-factors $g(2^+)$ of ^{106}Cd and ^{106}Pd with the available experimental data. The same HFB wave functions are employed to calculate the NTME $M_{2\nu}$ and the half-life $T_{1/2}^{2\nu}$ of ^{106}Cd for the 2ν $\beta^+\beta^+$, 2ν β^+EC and 2ν ECEC modes. The values of $T_{1/2}^{2\nu}$ calculated in the PHFB model with the summation method are larger than the previous calculations. The presently calculated NTME $M_{2\nu}$ is smaller than the recently given results in the QRPA(WS) model of Suhonen and Civitarese [25] by a factor of 2 approximately for all the three modes. The proton-neutron part of the PPQQ interaction that reflects the deformations of the intrinsic ground state, plays an important role in the quenching of $M_{2\nu}$ by a factor of 2 approximately in this particular case. The calculated 2ν e^+DBD decay half-lives are very close to the experimentally observable limits for the 2ν β^+EC and 2ν ECEC modes. It is hoped that the calculated $T_{1/2}^{2\nu}$, which is of the order of 10^{21-23} y can be reached experimentally for the 2ν β^+EC mode in the near future [14].

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